International Research Journal of Education and Technology
Peer Reviewed Journal
ISSN 2581-7795

ON FINDING INTEGER SOLUTIONS TO THE NON-HOMOGENEOUS TERNARY QUINTIC DIOPHANTINE EQUATION

$$x^2 + y^2 - xy = 28z^5$$

J. Shanthi[1], M. Parkavi[2]

[1]Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.

Email id: shanthivishvaa@gmail.com

[2]PG Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.

Email id: parkavimathiyalagan@gmail.com

Abstract:

This paper concerns with the problem of obtaining non-zero distinct integer solutions to the non-homogeneous Quintic Diophantine equation with three unknowns given by $x^2 + y^2 - xy = 28z^5$. Various sets of distinct integer solutions to the considered quintic equation are studied through employing the linear transformations $x = u + v, y = u - v, (u \neq v \neq 0)$ and applying the method of factorization.

Keywords: Ternary quintic, Non-Homogeneous quintic, Integer solutions.

INTRODUCTION

The non-homogeneous ternary Quintic Diophantine equation offers an unlimited field for research due to their variety [1-4]. In particular, one may refer [5-16] for quintic equations with three and five unknowns. This communication concerns with yet another interesting ternary Quintic equation $x^2 + y^2 - xy = 28z^5$ is analysed for its non-zero distinct integer solutions through different methods.

METHODS OF ANALYSIS

The non-homogeneous ternary Quintic Diophantine equation to be solved for non-zero distinct integral solution is

$$x^2 + y^2 - xy = 28z^5 \tag{1}$$

To start with, observe that (1) is satisfied by the following integer triples (x, y, z) : $(6,4,1)$, $(-6,-4,1)$, $(4,6,1)$,$(-4,-6,1)$, $(6\alpha^{5k}, 2\alpha^{5k}, \alpha^{2k})$, $\left(2.7^3.k(k^2-k+1)^2, 2.7^3(k^2-k+1)^2, 7(k^2-k+1)\right)$

However, there are other sets of integer solutions to (1) that are illustrated below:

ILLUSTRATION 1:

Introduction of the linear transformations

$$x = u + v, \quad y = u - v, \quad u \neq v \neq 0 \tag{2}$$

In (1) leads to

$$u^2 + 3v^2 = 28z^5 \tag{3}$$

The above equation is solved for u, v and z through different methods and using (2), the values of x and y satisfying (1), are obtained which are illustrated below

Method I:

After performing a few calculations, it is observed that (3) is satisfied by,

$$\begin{aligned} u &= 28^3 m(m^2 + 3n^2)^2 \\ v &= 28^3 n(m^2 + 3n^2)^2 \\ z &= 28(m^2 + 3n^2) \end{aligned} \tag{4}$$

In view of (2), The corresponding integer solutions to (1) are found to be

$$x = 28^3\left[m^2 + 3n^2\right]^2(m+n)$$
$$y = 28^3\left[m^2 + 3n^2\right]^2(m-n)$$
$$z = 28\left[m^2 + 3n^2\right]$$

Method II:

Assume

$$z = a^2 + 3b^2 \quad (5)$$

Case (i):

Write 28 as

$$28 = (5 + i\sqrt{3})(5 - i\sqrt{3}) \quad (6)$$

Using (5) and (6) in (3) and employing the method of factorization, define

$$(u + i\sqrt{3}v) = (5 + i\sqrt{3})(a + i\sqrt{3}b)^5$$

Equating the real and imaginary parts, we get

$$u = 5a^5 - 15a^4b + 225ab^4 - 150a^3b^2 + 90a^2b^3 - 27b^5$$

$$v = a^5 + 25a^4b + 45ab^4 - 30a^3b^2 - 150a^2b^3 + 45b^5$$

In view of (2), we obtain

$$x = 6a^5 + 10a^4b + 270ab^4 - 180a^3b^2 - 60a^2b^3 + 18b^5$$
$$y = 4a^5 - 40a^4b + 180ab^4 - 120a^3b^2 + 240a^2b^3 - 72b^5 \quad (7)$$

Thus (5) and (7) represent the integer solution to (1).

Case (ii):

Write 28 as

$$28 = (1 + i3\sqrt{3})(1 - i3\sqrt{3}) \quad (8)$$

Using (5) and (8) in (3) and employing the method of factorization, define

$$(u+i\sqrt{3}v)=(1+i3\sqrt{3})(a+i\sqrt{3}b)^5$$

Equating the real and imaginary parts, we get

$$u=a^5-45a^4b+45ab^4-30a^3b^2+270a^2b^3-81b^5$$
$$v=3a^5+5a^4b+135ab^4-90a^3b^2-30a^2b^3+9b^5$$

In view of (2), we obtain

$$\begin{aligned} x&=6a^5-40a^4b+180ab^4-120a^3b^2+240a^2b^3-72b^5 \\ y&=-2a^5-50a^4b-90ab^4+60a^3b^2+300a^2b^3-90b^5 \end{aligned} \tag{9}$$

Thus (5) and (9) represent the integer solution to (1).

Method III:

Equation (3) can be written as

$$u^2+3v^2=28z^5*1 \tag{10}$$

Write 1 on the R.H.S. of (10) as

$$1=\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{2^2} \tag{11}$$

Using (5), (6) & (11) in (10) and utilizing the method of factorization, define

$$(u+i\sqrt{3}v)=(5+i3\sqrt{3})(a+i\sqrt{3}b)^5\left[\frac{(1+i\sqrt{3})}{2}\right]$$

Equating the real and imaginary parts, the values of u and v are obtained as

$$u=\frac{1}{2}[2a^5-90a^4b+90ab^4-60a^3b^2+540a^2b^3-162b^5]$$

$$v=\frac{1}{2}[6a^5+10a^4b+270ab^4-180a^3b^2-60a^2b^3+18b^5$$

In view of (2), we obtain

$$\begin{aligned} x&=4a^5-40a^4b+180ab^4-120a^3b^2+240a^2b^3-72b^5 \\ y&=-2a^5-50a^4b-90ab^4+60a^3b^2+300a^2b^3-90b^5 \end{aligned} \qquad (12)$$

Thus (5) and (12) represent the integer solution to (1).

NOTE 1:

The integer 1 on the R.H.S of (10) is also expressed as below:

i) $1=\frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{7^2}$

ii) $1=\frac{(3r^2-s^2++i\,2\sqrt{3}rs)(3r^2-s^2-i\,2\sqrt{3}rs)}{(3r^2+s^2)^2}$

By considering suitable combinations of integers 28 & 1 from Note 1 in (3), some more sets of integer solutions to (1) are obtained.

ILLUSTRATION II:

Introduction of the linear transformations

$$x=2(u+v),\quad y=2(u-v),\quad u\neq v\neq 0 \qquad (13)$$

in (1) leads to

$$u^2+3v^2=7z^5 \qquad (14)$$

The above equation is solved for u, v and z through different methods and using (13), the values of x and y satisfying (1), are obtained which are illustrated below

Method IV:

After some algebra, it is seen that (14) is satisfied by,

$$\begin{aligned} u &= 7^3 m(m^2+3n^2)^2 \\ v &= 7^3 n(m^2+3n^2)^2 \\ z &= 7(m^2+3n^2) \end{aligned} \tag{15}$$

In view of (13), The corresponding integer solutions to (1) are found to be

$$\begin{aligned} x &= 2.7^3[m^2+3n^2]^2(m+n) \\ y &= 2.7^3[m^2+3n^2]^2(m-n) \\ z &= 7[m^2+3n^2] \end{aligned}$$

Method V:

Write 7 as

$$7 = (2+i\sqrt{3})(2-i\sqrt{3}) \tag{16}$$

Using (5) and (16) in (14) and employing the method of factorization, define

$$(u+i\sqrt{3}v) = (2+i\sqrt{3})(a+i\sqrt{3}b)^5$$

Equating the real and imaginary parts, we get

$$\begin{aligned} u &= 2a^5 - 15a^4b + 90ab^4 - 60a^3b^2 + 90a^2b^3 - 27b^5 \\ v &= a^5 + 10a^4b + 45ab^4 - 30a^3b^2 - 60a^2b^3 + 18b^5 \end{aligned}$$

In view of (13), we obtain

$$\begin{aligned} x &= 3a^5 - 5a^4b + 135ab^4 - 90a^3b^2 + 30a^2b^3 - 9b^5 \\ y &= a^5 - 25a^4b + 45ab^4 - 30a^3b^2 + 150a^2b^3 - 45b^5 \end{aligned} \tag{17}$$

Thus (5) and (17) represent the integer solution to (1).

Method III:

Equation (14) can be written as

$$u^2 + 3v^2 = 7z^5 * 1 \tag{18}$$

Using (5), (16) & (11) in (18) and utilizing the method of factorization, define

$$(u + i\sqrt{3}v) = (2 + i\sqrt{3})(a + i\sqrt{3}b)^5 \left[\frac{(1 + i\sqrt{3})}{2}\right]$$

Equating the real and imaginary parts, the values of u and v are obtained as

$$u = \frac{1}{2}[-a^5 - 45a^4b - 45ab^4 + 30a^3b^2 + 270a^2b^3 - 81b^5]$$

$$v = \frac{1}{2}[3a^5 - 5a^4b + 135ab^4 - 90a^3b^2 + 30a^2b^3 - 9b^5$$

In view of (13), we obtain

$$\begin{aligned} x &= a^5 - 25a^4b + 45ab^4 - 30a^3b^2 + 150a^2b^3 - 45b^5 \\ y &= -2a^5 - 20a^4b - 90ab^4 + 60a^3b^2 + 120a^2b^3 - 36b^5 \end{aligned} \tag{19}$$

Thus (5) and (19) represent the integer solution to (1).

CONCLUSION:

In this paper, we have presented three different methods of obtaining infinitely many non-zero distinct integer solutions of the non-homogeneous given by $x^2 + y^2 - xy = 28z^5$. To conclude, one may search for integer solutions to the other choices of non-homogeneous ternary quintic diaphonic equations along with suitable properties.

REFERENCES

[1] Dickson, L.E. 1952. History of Theory of Numbers, Vol.11, Chelsea Publishing company, New York.

[2] Mordell, L.J. 1969. Diophantine equations, Academic Press, London.

[3] Telang, S.G. 1996. Number theory, Tata Mc Graw Hill publishing company, New Delhi .

[4] Carmichael, R.D. 1959. The theory of numbers and Diophantine Analysis, Dover publications, New York.

[5] M. A. Gopalan, S. Vidhyalakshmi, J. Shanthi, " The non-homogeneous quintic equations with five unknowns $x^4 - y^4 + 2k(x^2 + y^2)(x - y + k) = (a^2 + b^2)(z^2 - w^2)p^{3n}$", open journal of applied and theoretical mathematics, vol-2, no 3, 08-13, sep 2016.

[6] S. Vidhyalakshmi, T. Mahalakshmi, V. Anbuveli, M. A. Gopalan, "On non-homogeneous quintic equation with five unknowns $3(x^4 y^4) = 4(z^2 - w^2)T^3$, Vidyabharati International interdisciplinary research journal, special issue on recent research trends in management, science and technology 1969-1972, August2021.

[7] M.A.Gopalan., and A. Vijsyashankar., Integral solutions of ternary quintic Diophantine equation $x^2 + (2k+1)y^2 = z^5$ International Journal of Mathematical and Sciences, 19(1-2), 165-169,2010.

[8] M.A.Gopalan., G. Sumathi and S. Vidhyalakshmi., Integral solutions of non-homogeneous ternary quintic equation in terms of pell equation JAMS, vol-6, No-1, 56-62,2013.

[9] M.A. Gopalan., S. Vidhyalakshmi., and K.Lakshmi., "Integral Solutions of the non-homogeneous equation ternary quadratic Diophantine equation $ax^2 + by^2 = (a+b)z^5, a > b > 0$", Archimedes Journal of Mathematics, vol-3,No-2, 197-204, 2013.

[10] M.A. Gopalan and G. Sangeetha, " Integral Solutions of ternary quadratic Diophantine equation $x^2 + y^2 = (k^2 + 1)z^5$",Bulletin of Pure and Applied Sciences, vol-29, No-1, 23-28, 2010.

[11] M.A.Gopalan., S. Vidhyalakshmi., and A. Kavitha, "Observations on the quintic equation with four unknowns $(x^3 + y^3)(x^2 + xy + y^2) + (x + y)(x^2 + y^2)w^2 = z(x^4 + x^3 y + x^2 y^2 + xy^3 + y^4)$", Impact J. Sci. Tech., Vol.7, No-2, 15-19,2013.

[12] M.A.Gopalan., S. Vidhyalakshmi., and A. Kavitha, "On the quintic equation with four unknowns $x^4 - y^4 = 2(k^2 + l^2)z^4 w$", Bessel J.Math., 3(2), 187-193,2013.

[13] M.A.Gopalan., S. Vidhyalakshmi., A. Kavitha., and E.Premalatha, "On the quintic equation with five unknowns $x^3 - y^3 = z^3 - w^3 + 6t^5$", International Journal of Current Research, vol-5, Iss:6,1437-1440, 2013.

[14] M.A.Gopalan., S. Vidhyalakshmi., and A. Kavitha., "On the quintic equation with five unknowns $2(x - y)(x^3 + y^3) = 19(z^2 - w^2)P^3$", International Journal of Engineering Research Online, vol-1, Iss:2, 2013.

[15] M.A.Gopalan., V.Krithika., and A.Vijayasankar, " Integral solutions of non-homogeneous quintic equation with five unknowns $3(x^4 - y^4) = 26(z^2 - w^2)P^3$", International Journal of Research – GRANTHAALAYAH,Vol-5, Iss:8, 25-33, 2017.

[16] S.Vidhyalakshmi, T. Mahalakshmi, G. Dhanalakshmi, M.A. Gopalan, On The Non-Homogeneous Quintic Equation with Five Unknowns $(x^4 - y^4) = 13P^3(z^2 - w^2)$, IAR J Eng Tech,1(1), 43-49,2020.